Pulse velocity in a granular chain

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We discuss the applicability of two very different analytic approaches to the study of pulse propagation in a chain of particles interacting via a Hertz potential, namely, a continuum model and a binary collision approximation. While both methods capture some qualitative features equally well, the first is quantitatively good for softer potentials and the latter is better for harder potentials.

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The physics of a chain of particles interacting via a granular potential, i.e., a potential that is repulsive under loading and zero otherwise, remains a challenge despite a great deal of recent work on the subject [1–15]. Theoretical studies of pulse dynamics in frictionless chains have relied primarily on numerical solution of the equations of motion [2,5,7–9,13,16]. Analytic work has relied on two rather different, in some sense contradictory, approximations, with very little direct comparison between them. It is our purpose to compare the predictions of these two approaches for the pulse velocity to each other and to numerical simulations, in order to assess the regimes of validity of each.

One of the approaches is based on continuum approximations to the equations of motion followed by exact or approximate solutions of these equations [7,16–18]. This approach is expected to give useful results when the pulse is rather *broad*, i.e., when the velocity distribution of the grains in the chain at any instant of time is rather smooth. The other approach is based on phenomenology about properties of pairwise (or at times three-body) collisions together with the assumption that pulses are sufficiently *narrow* to involve only two or three grains at any one time [14,19,20]. Some rigorous methods in the literature might, with additional work, also be brought to bear on the problem [12,21]. Among the interesting quantities that one aims to calculate with these approaches is the pulse velocity. In turn, successful calculation of the pulse velocity requires a good understanding of the pulse width.

The standard generic model potential between monodisperse elastic granules that repel upon overlap according to the Hertz law is given by [22,23]

$$V(\delta_{k,k+1}) = \frac{a}{n}|\delta|_{k,k+1}^{n}, \qquad \delta \leqslant 0,$$

$$V(\delta_{k,k+1}) = 0, \qquad \delta > 0. \tag{1}$$

Here

$$\delta_{k,k+1} \equiv y_{k+1} - y_k, \tag{2}$$

a is a constant that depends on the Young's modulus and Poisson's ratio, and y_k is the displacement of granule k from its equilibrium position. The exponent n is 5/2 for spheres, it is 2 for cylinders, and in general depends on geometry.

The displacement of the kth granule ($k=1,2,\ldots,L$) in the chain from its equilibrium position in a frictionless medium is thus governed by the equation of motion

$$m\frac{d^2y_k}{d\tau^2} = -a(y_k - y_{k+1})^{n-1}\theta(y_k - y_{k+1}) + a(y_{k-1} - y_k)^{n-1}\theta(y_{k-1} - y_k). \tag{3}$$

Here $\theta(y)$ is the Heaviside function, $\theta(y)=1$ for $y>0$, $\theta(y)=0$ for $y<0$, and $\theta(0)=1/2$. It ensures that the particles interact only when in contact. In this report we consider a finite open chain, and therefore the first term on the right-hand side of this equation is absent for the last granule and the second term is absent for the first.

Initially the granules are placed along a line so that they just touch their neighbors in their equilibrium positions (no precompression), and all but the leftmost particle are at rest. The initial velocity of the leftmost particle is v_0 (the impulse). In terms of the rescaled variables

$$y_k = \left(\frac{mv_0^2}{a}\right)^{1/n} x_k, \qquad \tau = \frac{1}{v_0}\left(\frac{mv_0^2}{a}\right)^{1/n} t, \tag{4}$$

Eq. (3) can be rewritten as

$$\ddot{x}_k = -(x_k - x_{k+1})^{n-1}\theta(x_k - x_{k+1}) + (x_{k-1} - x_k)^{n-1}\theta(x_{k-1} - x_k), \tag{5}$$

where a dot denotes a derivative with respect to t. In the rescaled variables the initial conditions become $x_k(0) = \dot{x}_k(0) = 0$, $\forall k \neq 1$, $x_1(0)=0$, and $\dot{x}_1(0)=1$.

When $n>2$ an initial impulse settles into a pulse that is increasingly narrow with increasing n, and propagates at a velocity that is essentially constant and determined by n and by the amplitude of the pulse. For $n=2$ the pulse spreads in time and travels at a constant velocity independent of pulse amplitude. In the latter case there is considerable backscattering that leads to backward motion of all the granules behind the pulse, whereas the backscattering is minimal for $n>2$ [7,24]. In this case the pulse is a solitary wave [12] that becomes narrower and narrower as n increases.

Three features determine pulse dynamics in these chains: (1) the power n in the potential; (2) the absence of a restoring

force; and (3) the discreteness of the system. Recently, we discussed the role each of these features in the continuum approximation, and extended previous results to include viscosity [24]. Not only does this approximation work very well for the $n=2$ case [7,24], but we confirmed a result well known in the field, namely, that the continuum approximation works surprisingly well for the prototypical spherical grains [7,17,24]. However, a discussion of the reasons for this agreement seems to be lacking. On the other hand, at the other approximative extreme, models based on binary interactions have also been proposed in order to study a chain of spherical and other grains. Wu's independent-collision model [14] focuses on a chain of tapered grains. From energy and momentum conservation considerations, working in the $n \to \infty$ limit, he shows that his simple model captures the qualitative behavior observed in simulations for spheres. Subsequently this model was phenomenologically extended and compared with experimental results [15]. We thus see that two somewhat contrary approximations seem to work rather well for spherical grains ($n=5/2$), but it is not clear which (or why) works better, nor is it clear how well each works with changing n.

Herein we address the question of the applicability of both the continuum approximation and a binary interaction model through the analysis of the velocity of the signal propagation as a function of the power of the potential. For the former case, the pulse velocity, $C_c(n)$, can explicitly be written as [24]

$$C_c(n)=\sqrt{\frac{2}{n}}\left[\frac{n^2(n-2)}{2(n+2)\sqrt{\frac{n(n-1)}{6}}I\left(\frac{4}{n-2}\right)}\right]^{(n-2)/2n}, \tag{6}$$

where

$$I(l)=2^l\frac{\Gamma^2\left(\frac{l+1}{2}\right)}{\Gamma(l+1)} \tag{7}$$

(the quantity $C_c(n)$ here is the same as c_0 in Ref. [24]].

On the other hand, for the binary collision approximation the set of equations (5) reduces to two coupled equations which may be decoupled by defining the normal mode variables $z_\pm = x_1 \pm x_2$. In particular, we have

$$\ddot{z}_- = -2z_-^{n-1}. \tag{8}$$

This is precisely the equation of motion for one particle subjected to a potential $V(z)=2z^n/n$. Furthermore, the initial conditions for the original variables lead to $z_-(0)=0$ and $\dot{z}_-(0)=1$. Hence, from energy conservation we have

$$\frac{1}{2}\dot{z}_-^2(t)-\frac{1}{2}=-\frac{2}{n}z_-^n(t). \tag{9}$$

Consequently, the time at which the two particles have the same velocity ($\dot{z}_-=0$) is the time of maximum compression, given by $z_m=(n/4)^{1/n}$. Therefore, the time necessary for the pulse to travel from the first to the second particle is

$$T_b(n)=\int_0^{z_m}\frac{1}{\sqrt{1-4z_-^n/n}}\mathrm{d}z_-. \tag{10}$$

Explicit integration of Eq. (10) then leads to the pulse velocity $C_b(n)=1/T_b(n)$,

$$C_b(n)=\frac{1}{\pi^{1/2}}\left(\frac{4}{n}\right)^{1/n}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}{\Gamma\left(1+\frac{1}{n}\right)}. \tag{11}$$

Our comparison is thus between Eqs. (6) and (11).

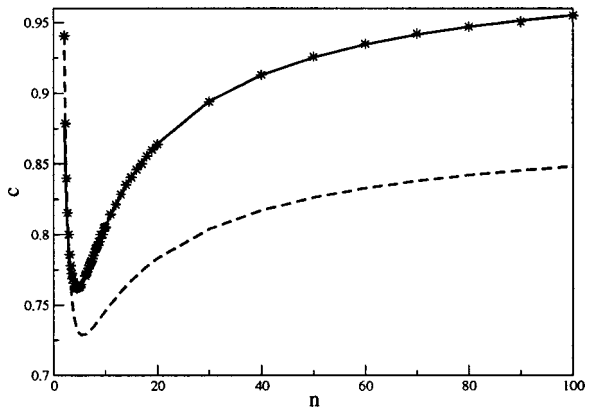

FIG. 1. Pulse velocity as a function of the power of the potential. The stars represent the numerical simulation results, the continuous line represents the binary collision approximation, and the broken line represents the continuum approximation.

Qualitatively both approximations give the same result: for $n \gtrsim 2$ the pulse velocity decreases with n, attaining its minimum value for $n \simeq 5$, and then increasing and saturating for large n (see Fig. 1). Quantitatively, however, they differ appreciably. For instance, for large n, $C_b \to 1$ while $C_c \to 0.883\ldots$.

In Fig. 2 we present the relative error of our numerical simulation of the chain as compared with each theory. These simulations were performed in chains with up to 5000 grains using a fifth order gear algorithm. The pulse velocity was determined as the slope of the curve $k_{max}(t)$, where $k_{max}(t)$ is the grain with maximum velocity at time t. Moreover, it was observed that the pulse settles into its final shape and a constant velocity after traveling along just a few grains. From this figure, it is easy to see that while the continuum approximation gives very good results for small n, its predictions are poor for $n \gtrsim 3.0$. On the other hand, the binary collision approximation is extremely accurate for $n \gtrsim 3.0$ but not accurate for small n. The quantitative agreement of each of the two analytic results at their respective n extremes with the numerical simulations is seen to be excellent. Again, we stress that *these results are a reflection of the behavior of the pulse width*. At smaller n the pulse is relatively broad, the velocity pulse covers a number of grains [7,24], and a continuum approximation captures the pulse configuration and

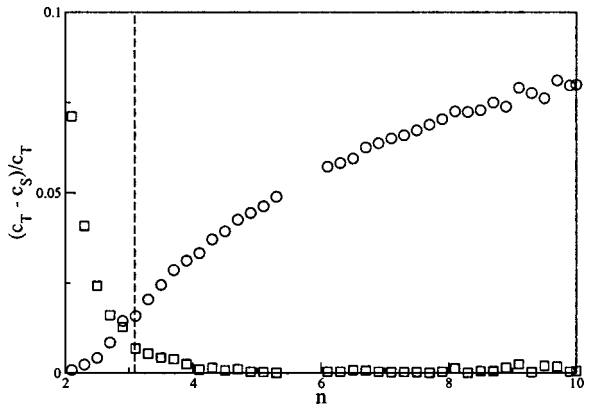

FIG. 2. Relative error $(C_T - C_S)/C_T$ in the pulse velocity as a function of the power of the potential. Here, the theoretical pulse velocity, C_T, is either the one obtained from the continuum (circles) or from the binary collision (squares) approximation. C_S is the pulse velocity obtained from the numerical simulations of the chain. The dashed line is at $n=3.084$, corresponding to $\alpha=3$ in Eq. (12).

speed very well. At larger n the pulse becomes very narrow, discreteness effects dominate the behavior, and an approximation that assumes that two-grain collisions dominate the pulse behavior reproduces the pulse velocity extremely accurately. To further support this conclusion, one can explicitly calculate the pulse width α as a function of n in the continuous approximation [7,17,24]:

$$\alpha=\left[\frac{6(n-2)^2}{n(n-1)}\right]^{1/2}. \tag{12}$$

The resulting width is less than three grains when $n>3.084$.

In summary, continuum approximations give quantitatively accurate results for the pulse velocity in discrete granular chains with relatively soft interactions, $n\lesssim 3.0$ (which includes the generic cases of cylindrical and spherical grains). We have explicitly shown this on the basis of Nesterenko's theory, but improvements thereof do not appreciably increase the validity of continuum theories beyond $n\sim 3$. This is of course connected to the fact that beyond this n, the pulse is too narrow for continuum theories. The binary collision model is quantitatively correct for relatively hard potentials, $n\gtrsim 3.0$, the regime where the pulse involves essentially no more than two granules at a time. The interesting point is that the pulse velocity for the entire range of potentials can be obtained with quantitative accuracy with one or the other of these two limiting approaches, with a maximum relative error of about 2% at a value of the exponent near that of the ubiquitous spherical grains.

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